A SU(3) Octet Study on the Effects of Meson-Baryon Vertices on Baryon-Baryon Interactions

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Motivation

Nuclear Physics Goals

ELPH          DAΦNE
HADES         GSI
AMADEUS       FAIR
J-PARC        PANDA
J-LAB         WASA etc.
Motivation

Nuclear Physics Goals

Baryon-Baryon Interaction

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Motivation

Nuclear Physics Goals

- ELPH
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- PANDA
- J-LAB
- WASA etc.

Baryon-Baryon Interaction

Meson- Baryon Interactions
A SU(3) Octet Study of effect of Meson-Baryon Vertices on Hyperon-Baryon Interactions

- YB Interaction model
  - OBE
  - SU(3) à la Quark Model

- Scattering equation

- Results
  - effect of MB vertex on YB
Different Methods

- **Quantum Chromo Dynamics** *(NPLQCD and HALQCD)*
- Chiral EFT *(e.g. J. Haidenbauer (Jülich))*
- Extension from the Nucleon sector
- Meson Exchange Potential *(Nijmegen, Jülich etc.)*
- Quark-cluster model *(Oka et al., Fujiwara et al.)*

**Deciding factor**
Description of short and long range behaviour
How to proceed??

- Quantum Chromo Dynamics (NPLOQCD and HALQCD)
- Chiral EFT (e.g. J. Haidenbauer)
- Extension from the Nucleon sector
- Meson Exchange Potential (Nijmegen, Jülich etc.)
- Quark-cluster model (Oka et al., Fujiwara et al.)

Our model

Short and long range by ONLY One Meson Exchange

Duration factor

Description of short and long range behaviour
Model Description
- Interaction Lagrangian of the interaction \((x=ps,s,v)\)

\[
\mathcal{L}^{x}_{int} = -g_x \alpha_x \text{Tr}([B, \bar{B}] \phi_x) + g_x (1 - \alpha_x) \text{Tr}([\bar{B}, B] \phi_x)
\]

- Mesons are mediators
Interaction model

- Interaction Lagrangian of the interaction \((x=ps,s,v)\)

\[
L_{int}^x = -g_x \alpha_x Tr\left([B, \bar{B}]\phi_x\right) + g_x (1 - \alpha_x) Tr\left([\bar{B}, B]\phi_x\right)
\]

- Mesons are mediators

\[
V_{ps}\left(\vec{k}, \vec{p}\right) = -\frac{g_{ps}^2}{4M^2} \frac{\left(\sigma_1 \cdot \vec{k}\right)\left(\sigma_2 \cdot \vec{k}\right)}{k^2 + m_{ps}^2}
\]

\[
V_{s}\left(\vec{k}, \vec{p}\right) = -\frac{g_s^2}{2M^2} \frac{1}{k^2 + m_s^2} \left[2M^2 - p^2 + \frac{k^2}{4} - i S \cdot (\vec{k} \times \vec{p})\right]
\]

\[
V_v\left(\vec{k}, \vec{p}\right) = \frac{1}{k^2 + m_v^2} \left(\frac{g_v^2}{2M^2} \left[2M^2 + 3p^2 - \frac{k^2}{4} + 3i S \cdot (\vec{k} \times \vec{p}) - \sigma_1 \cdot \sigma_2 \frac{k^2}{2} + \frac{1}{2} \left(\sigma_1 \cdot \vec{k}\right)\left(\sigma_2 \cdot \vec{k}\right)\right]
\]

\[
+ \frac{g_v f_v}{2M^2} \left[-k^2 + 4i S \cdot (\vec{k} \times \vec{p}) - \sigma_1 \cdot \sigma_2 k^2 + \left(\sigma_1 \cdot \vec{k}\right)\left(\sigma_2 \cdot \vec{k}\right)\right]
\]

\[
+ \frac{f_v^2}{4M^2} \left[-\sigma_1 \cdot \sigma_2 k^2 + \left(\sigma_1 \cdot \vec{k}\right)\left(\sigma_2 \cdot \vec{k}\right)\right]\]

(3.10)
Interaction model

- Interaction Lagrangian of the interaction \((x=ps,s,v)\)

\[
\mathcal{L}_{int}^x = -g_x \alpha_x Tr([B, \bar{B}] \phi_x) + g_x (1 - \alpha_x) Tr([\bar{B}, B] \phi_x)
\]

- Mesons are mediators

- Caveat: How to find couplings?
Interaction model

- Interaction Lagrangian of the interaction \((x=ps,s,v)\)

\[
\mathcal{L}_{int}^x = -g_x \alpha_x Tr([B, \bar{B}]\phi_x) + g_x(1 - \alpha_x) Tr([\bar{B}, B]\phi_x)
\]

- Mesons are mediators

- Caveat: How to find couplings?

- Possibilities: 1. Fit to data
  2. Measure directly from MB scattering expt
  3. Theoretical calculations for e.g. reaction, scattering
  4. QCD sum rule
  5. Underlying symmetry \(\rightarrow SU(3)\) via Quark Model
- Quark flavour symmetry → The Eightfold Way

- Gell-Mann and Zweig: Patterns could be explained if all hadrons were made of quarks

$$B(qqq), \Phi(q\bar{q})$$
Group Theory Interpretation :: The Quark Model

**Mesons**

\[ M(q\bar{q}) = 3 \otimes \bar{3} = 8 \oplus 1 \]

**Baryons**

\[ B(qqq) = 3 \otimes 3 \otimes 3 = (3 \otimes 6) \oplus (3 \otimes \bar{3}) = 10_S \oplus 8_M \oplus 8_M \oplus 1_A \]
Flavour Symmetry

- SU(3) algebra is given by \[ \left[ \begin{array}{c} \lambda_a \\ \lambda_b \\ \frac{\lambda_c}{2} \end{array} \right] = i f_{abc} \frac{\lambda_c}{2} \], \( \lambda \): fundamental matrices of SU(3)

- Baryons and Meson octets in matrix forms which are SU(3) invariant

\[
B = \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \lambda^a B^a = \begin{pmatrix}
\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\
-\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Xi^0 & n \\
\Xi^- & -\frac{2\Lambda}{\sqrt{6}} & \Xi^0
\end{pmatrix}
\]

\[
\phi_{ps} = \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \lambda^a \phi_{ps}^a = \begin{pmatrix}
\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\
-\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 & 2\eta_8 \\
K^- & \bar{K}^0 & \bar{K}^0
\end{pmatrix}
\]

\[
\phi_v = \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \lambda^a \phi_v^a = \begin{pmatrix}
\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{6}} & \rho^+ & K^{*+} \\
-\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{6}} & K^{*0} & 2\omega \\
K^{*-} & \bar{K}^{*0} & \bar{K}^{*0}
\end{pmatrix}
\]

\[
\phi_s = \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \lambda^a \phi_s^a = \begin{pmatrix}
\frac{a_0^0}{\sqrt{2}} + \frac{f_0}{\sqrt{6}} & a_0^+ & \kappa^+ \\
-a_0^- & \frac{a_0^0}{\sqrt{2}} + \frac{f_0}{\sqrt{6}} & \kappa^0 \\
\kappa^- & \bar{\kappa}^0 & \bar{\kappa}^0
\end{pmatrix}
\]
Parameters of the Model

\[ g_{NN\pi} = g_8 \]
\[ g_{\Lambda\Sigma\pi} = \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Sigma\Sigma\pi} = 2 g_8 \alpha_{ps} \]
\[ g_{\Xi\Xi\pi} = -g_8 (1 - 2 \alpha_{ps}) \]
\[ g_{NN\eta_8} = \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{NN\eta_8} = \frac{1}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Xi\Xi\eta_8} = -\frac{1}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Xi\Xi\eta_8} = \frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Xi\Xi\eta_8} = -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Lambda\Lambda\eta_8} = \frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Xi\Xi\eta_8} = -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]

Example: Pseudo-scalar Meson – Baryon Couplings
Parameters of the Model

\[
\begin{align*}
g_{NN\pi} &= g_8 \\
g_{\Lambda\Sigma\pi} &= \frac{2}{\sqrt{3}} g_8 (1-\alpha_{ps}) \\
g_{\Sigma\Sigma\pi} &= 2 g_8 \alpha_{ps} \\
g_{\Xi\Xi\pi} &= -g_8 (1-2\alpha_{ps}) \\
g_{NN\eta_1} &= g_8 \\
g_{\Lambda\Lambda\eta_1} &= g_{\Sigma\Sigma\eta_1} \\
g_{\Xi\Xi\eta_1} &= -g_8 \\
g_{\Lambda\Lambda\eta_s} &= g_{\Sigma\Sigma\eta_s} = \frac{2}{\sqrt{3}} g_8 (1-\alpha_{ps}) \\
g_{\Xi\Xi\eta_s} &= \frac{1}{\sqrt{3}} g_8 (1+2\alpha_{ps}) \\
g_{\Xi\Sigma\eta_s} &= -\frac{1}{\sqrt{3}} g_8 (1+2\alpha_{ps}) \\
g_{\Lambda\Lambda\eta_s} &= g_{\Sigma\Sigma\eta_s} = \frac{2}{\sqrt{3}} g_8 (1-\alpha_{ps}) \\
g_{\Xi\Xi\eta_s} &= \frac{1}{\sqrt{3}} g_8 (1+2\alpha_{ps}) \\
g_{\Xi\Sigma\eta_s} &= -\frac{1}{\sqrt{3}} g_8 (1+2\alpha_{ps}) \\
g_{\Lambda\Lambda\eta_1} &= g_{\Sigma\Sigma\eta_1} = g_{\Xi\Xi\eta_1} = g_8 \\
g_{NN\eta_1} &= \cos(\theta_{ps}) g_{NN\eta_s} - \sin(\theta_{ps}) g_{NN\eta_1}
\end{align*}
\]

Example: Pseudo-scalar Meson – Baryon Couplings

For singlet meson \( \eta_1 \):

\[
g_{NN\eta_1} = g_{\Lambda\Lambda\eta_1} = g_{\Sigma\Sigma\eta_1} \equiv g_1
\]

Physical \( \eta \) coupling:

\[
g_{NN\eta_1} = \cos(\theta_{ps}) g_{NN\eta_s} - \sin(\theta_{ps}) g_{NN\eta_1}
\]
Parameters of the Model

\[ g_{NN\pi} = g_8 \]
\[ g_{\Lambda\Sigma\pi} = \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Sigma\Sigma\pi} = 2 g_8 \alpha_{ps} \]
\[ g_{\Xi\Xi\pi} = -g_8 (1 - 2 \alpha_{ps}) \]
\[ g_{\Lambda\Lambda\eta_8} = \frac{-2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Sigma\Sigma\eta_8} = \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \]
\[ g_{\Xi\Xi\eta_8} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Xi\Sigma\eta_8} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Lambda\Sigma\eta_8} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Lambda\Lambda\eta_1} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
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\[ g_{\Xi\Xi\eta_1} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]
\[ g_{\Xi\Sigma\eta_1} = \frac{-1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \]

Example: Pseudo-scalar Meson – Baryon Couplings

For singlet meson \( \eta_1 \):
\[ g_{NN\eta_1} = g_{\Lambda\Lambda\eta_1} = g_{\Sigma\Sigma\eta_1} = g_1 \]

Physical \( \eta \) coupling:
\[ g_{NN\eta} = \cos(\theta_{ps}) g_{NN\eta_8} - \sin(\theta_{ps}) g_{NN\eta_1} \]
Parameters of the Model

\[
\begin{align*}
    g_{NN\pi} &= g_8 \\
    g_{\Lambda\Sigma\pi} &= \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \\
    g_{\Sigma\Sigma\pi} &= 2 g_8 \alpha_{ps} \\
    g_{\Xi\Xi\pi} &= -g_8 \\
    g_{NN\eta_8} &= \frac{2}{\sqrt{3}} g_8 (1 - \alpha_{ps}) \\
    g_{NN\eta_1} &= \cos(\theta_{ps}) g_{NN\eta_8} - \sin(\theta_{ps}) g_{NN\eta_1} \\
    g_{\Lambda\Lambda\eta_8} &= -2 \\
    g_{\Lambda\Lambda\eta_1} &= -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \\
    g_{\Sigma\Sigma\eta_8} &= \frac{1}{\sqrt{3}} g_8 (4 \alpha_{ps} - 1) \\
    g_{\Xi\Xi\eta_8} &= -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \\
    g_{\Xi\Xi\eta_1} &= -g_8 \\
    g_{\Lambda\Lambda\eta_8} &= -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \\
    g_{\Sigma\Sigma\eta_1} &= -\frac{1}{\sqrt{3}} g_8 (1 + 2 \alpha_{ps}) \\
\end{align*}
\]

Physical \( \eta \) coupling ::

Similar relations exist for scalar and vector mesons.
How to check the model?

→ Find observables
- 4D Bethe-Salpeter (BS) equation

\[ T = V + \int V G K \]

- Blankenbecler-Sugar reduction \( \Rightarrow \) 3DLippmann-Schwinger equation

- Solved in K-matrix formalism

\[ T = \frac{K}{1-iK}, \quad K = V + P \int V G K \]
How? Scattering Equation

- 4D Bethe-Salpeter (BS) equation

\[
T = \int V + V G T
\]

- Blankenbecler-Sugar reduction \(\Rightarrow\) 3DLippmann-Schwinger equation

- Solved in K-matrix formalism

\[
T = \frac{K}{1 - iK}, K = V + P \int V G K
\]

- In-medium effect: Multiply each Green function by the Pauli projector operator

\[
Q_F = \Theta \left( k_1^2 - k_F^2 \right) \quad \Rightarrow \quad K = V + P \int V Q_F G K
\]

Bethe-Goldstone Equation
Results
<table>
<thead>
<tr>
<th>Particle basis</th>
<th>Q= -2</th>
<th>Q= -1</th>
<th>Q= 0</th>
<th>Q= 1</th>
<th>Q= 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S=0</td>
<td></td>
<td></td>
<td><strong>nn</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S= -1</td>
<td></td>
<td><strong>Σ⁻n</strong></td>
<td></td>
<td><strong>Λn,Σ⁰n,Σ⁻p</strong></td>
<td><strong>Λp,Σ⁺n,Σ⁰p</strong></td>
</tr>
<tr>
<td>S= -2</td>
<td><strong>Σ⁻Σ⁻</strong></td>
<td><strong>Ξ⁻n,Σ⁻Λ,Σ⁻Σ⁰</strong></td>
<td><strong>ΛΛ,Ξ⁰n,Ξ⁻p, Σ⁰Λ,Σ⁰Σ⁰,Σ⁻Σ⁺</strong></td>
<td><strong>Ξ⁰p, Σ⁺Λ, Σ⁰Σ⁺</strong></td>
<td><strong>Σ⁺Σ⁺</strong></td>
</tr>
<tr>
<td>S= -3</td>
<td><strong>Ξ⁻Σ⁻</strong></td>
<td><strong>Ξ⁻Λ,Ξ⁰Σ⁻,Ξ⁻Σ⁰</strong></td>
<td><strong>Ξ⁰Λ, Ξ⁰Σ⁰,Ξ⁻Σ⁺</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S= -4</td>
<td><strong>Ξ⁻Ξ⁻</strong></td>
<td><strong>Ξ⁻Ξ⁰</strong></td>
<td></td>
<td><strong>Ξ⁰Ξ⁰</strong></td>
<td></td>
</tr>
</tbody>
</table>
Choosing the Parameters

<table>
<thead>
<tr>
<th></th>
<th>$g_8/\sqrt{4\pi}$</th>
<th>$g_1/\sqrt{4\pi}$</th>
<th>$\alpha$</th>
<th>$\theta_{\text{mixing}}$ (degree)</th>
<th>$\Lambda_c$(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>pseudoscalar</td>
<td>3.567- 3.795</td>
<td>2.08 - 4.16</td>
<td>.355-.491</td>
<td>-10 or -23 (Gell-mann Okubo Mass formula)</td>
<td>1.2-1.4</td>
</tr>
<tr>
<td>vector</td>
<td>.68-1.18</td>
<td>2.529-3.762</td>
<td>E:1</td>
<td>35.26(OZI Rule)</td>
<td>1.07-2</td>
</tr>
<tr>
<td>scalar</td>
<td>.76-1.395</td>
<td>3.17-4.598</td>
<td>.841-1.285</td>
<td>37.05 - 54.75</td>
<td>.988-2</td>
</tr>
</tbody>
</table>

**Starting point**

**Pseudoscalar:** $g_8/\sqrt{4\pi}=\sqrt{14}$ (Phenomenology)

$\alpha = .35$ (Cabbibo Theory of semi leptonic decays)

**Vector:**

$\alpha_E = 1$ (Universal coupling)

$\theta_{\text{mixing}} = 35.26$ degree

$\Lambda_c :$ ps = 1.3 GeV vector= 1.7 GeV scalar= 2 GeV
Effect of Vector Meson Coupling

$\Sigma^+ p \ {}^1S_0$ cross-section variation with vector meson coupling

![Graph showing $\Sigma^+ p \ {}^1S_0$ cross-section variation with vector meson coupling](image)

- Low $0.75$  
- High $1.2$

$\sigma$ [mb] vs $p_{Lab}$ [MeV/c]
Effect of Vector Meson Coupling

\[ \Sigma^+ p \quad {^1S_0} \] Phase-shift variation with vector meson coupling

\[
\begin{align*}
\omega_m &= 782.65 \\
\rho_m &= 775.26 \\
K^*_m &= 891.66
\end{align*}
\]
Effect of Scalar Meson Coupling

$\Sigma^+ p \ ^1S_0$ cross-section variation with scalar meson coupling

$g_s^8$ values:
- Low: 0.76
- High: 1.2

$\sigma$ [mb] vs. $p_{Lab}$ [MeV/c]
Effect of Scalar Meson Coupling

$\Sigma^+ p \to \text{vector meson}$

$\omega_m = 782.65$
$\rho_m = 775.26$
$K^*_m = 891.66$

$a_0 = 983.0d0$
$\epsilon = 760.0d0$
$\kappa = 880.0d0$

$\Sigma^+ p \to ^1S_0$ phase shift variation with scalar meson coupling
Nuclear Medium Effect
$\Sigma^+ p \ 1^S_0$ phase-shift variation with vector meson coupling
$\Sigma^+ p \, {}^1S_0$ phase-shift variation with scalar meson coupling
Effect of Cut-off

Multiplication of Form Factor to each BBM vertex to tackle divergence

\[
F = \left( \frac{\Lambda_c^2 - m^2}{\Lambda_c^2 + k^2} \right)^2
\]

\[\Sigma^+ p \ ^1S_0 \text{ phase-shift variation with cut-off}\]
Effect of Cut-off

Multiplication of **Form Factor** to each BBM vertex to tackle divergence

\[
\Sigma^+ p \ {^1S_0} \text{ phase-shift variation with cut-off}
\]

Cut-off plays important role in the BB interaction

\[
F = \left( \Lambda_c^2 - m^2 \right)^2 \left( \frac{m^2}{k^2 + m^2} \right)
\]

\[
\text{Low } 1.7 \text{ GeV}
\]

\[
\text{High } 2 \text{ GeV}
\]
$S=-2$  \[\Lambda\Lambda^1S_0\] phase-shift variation with scalar meson coupling
S=-2 $\Lambda\Lambda$ $^1S_0$ phase-shift variation with vector meson coupling
\[ \lim_{q \to 0} \frac{q}{\tan \delta} = q \cot \delta \approx -\frac{1}{a_s} + \frac{1}{2} r_s q^2 \]
$\lim_{q \to 0} \frac{q}{\tan \delta} = q \cot \delta \approx \frac{-1}{a_s} + \frac{1}{2} r_s q^2$
Conclusions

Summary

- YB interaction is sensitive to MB interaction in OBE model

- Knowledge about meson-baryon interaction is essential for the understanding of BB interaction

Outlook

- Inclusion of SU(3) breaking effect?
Thank you for your attention.
Fit to data for $S=-1$

$\Sigma^+p \leftrightarrow \Sigma^+p$

$\Lambda p \leftrightarrow \Lambda p$

Outcome

Relation between $k_F$ and $\rho$:

$$k_F = \sqrt[3]{3\pi^2} \rho$$

→ Free space cross-section decreases in presence of medium

(In preparation)
<table>
<thead>
<tr>
<th>Channels</th>
<th>Scattering Length ($a_s$) (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^+ p \ ^1S_0$</td>
<td>-1.76</td>
</tr>
<tr>
<td>$\Sigma^+ p \ ^3S_1$</td>
<td>1.03</td>
</tr>
<tr>
<td>$\Lambda p \ ^1S_0$</td>
<td>-2.46</td>
</tr>
<tr>
<td>$\Lambda p \ ^3S_1$</td>
<td>8.10</td>
</tr>
<tr>
<td>$\Lambda\Lambda \ ^1S_0$</td>
<td>-1.11</td>
</tr>
<tr>
<td>$\Lambda\Lambda \ ^3S_1$</td>
<td>-2.01</td>
</tr>
<tr>
<td>$\Sigma^+\Sigma^+ \ ^1S_0$</td>
<td>-0.38</td>
</tr>
<tr>
<td>$\Sigma^+\Sigma^+ \ ^3S_1$</td>
<td>0.47</td>
</tr>
</tbody>
</table>

- Our parameters:

\[
g_s^s/\sqrt{4\pi} = 0.95, \quad g_v^s/\sqrt{4\pi} = 1.178, \quad \alpha_s = 0.84,
\]

$\Lambda_c^s = 2.0$ GeV, $\Lambda_c^{ps} = 1.3$ GeV, $\Lambda_c^v = 1.7$ GeV
<table>
<thead>
<tr>
<th>Model Type</th>
<th>Short range</th>
<th>Medium, Long range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yukawa 1935</td>
<td>-</td>
<td>Pion (scalar)</td>
</tr>
<tr>
<td>Jülich 1989, 1994, 2004</td>
<td>OBE</td>
<td>OBE, $\Delta, 2\pi - K \bar{K}$</td>
</tr>
<tr>
<td>Arisaka et al. 2000</td>
<td>Gaussian</td>
<td>OBE</td>
</tr>
<tr>
<td>Z01X</td>
<td>Lattice QCD</td>
<td>OBE</td>
</tr>
<tr>
<td>Our model</td>
<td>OBE (ps, s, v)</td>
<td>OBE (ps, s, v)</td>
</tr>
</tbody>
</table>
ΛΛ → ΛΛ  phase-shifts
One way of solving the Lippmann-Schwinger equation is to use the Born series. However, in channels having a bound state \((^3S_1 + ^3D_1)\) or a resonance it is well known that the Born series diverges\(^4\). Also, for the case of a virtual state \((^1S_0)\) and for very strong potentials the Born series is known to converge slowly, if at all. Two methods that can be used have been suggested by Weinberg and by Noyes and Kowalski \(^4\).

A third and even simpler method is to use matrix inversion. Let us first consider (2.10) for uncoupled channels \((L = L')\) and add a zero term to replace the principal value condition by a smooth integrand

\[
R_L^\alpha(k|k_0) = V_L^\alpha(k|k_0) - \frac{2}{\pi} \int_0^\infty \frac{dk'[k'^2V_L^\alpha(k'|k_0)R_L^\alpha(k'|k_0) - k_0^2 V_L^\alpha(k|k_0)R_L^\alpha(k_0|k_0)]}{k'^2 - k_0^2}. \tag{2.16}
\]

The integrand has a finite limit even for \(k' = k_0\); however, we wish to avoid such points. Our goal is to solve eq. (2.16) numerically without having any points at which \(k' = k_0\). At the same time we need to find the \(R\)-matrix both on and off the energy shell.
These quantities can be easily found by introducing an $N$-point integration formula

$$\int_0^\infty dk F(k) = \sum_{j=1}^N F(k_j) \omega_j,$$

(2.17)

where we have preferred to take $k_j$ and $\omega_j$ to be either Laguerre or Gaussian integration points and weights. For Gaussian integration the mapping $k = tg_{\frac{1}{2}} \pi x$ was used. Gaussian integration is used for potentials having a relatively slow fall-off in momentum space.

All of the $N$ integration points, $k_1 \ldots k_N$, are required to be unequal to $k_0$. If we call $k_0$ the $N+1$ point ($k_0 \equiv k_{N+1}$), then eq. (2.16) can be rewritten as

$$V_L^\alpha(k_i|k_{N+1}) = \sum_{j=1}^{N+1} F_L^\alpha(k_i|k_j) R_L^\alpha(k_j|k_{N+1}).$$

(2.18)

The matrix $F_L$ is simply

$$V=R+\Sigma F \ R \quad \quad \quad F_L^\alpha(k_i|k_j) = \delta_{ij} + \omega_j^\prime V_L^\alpha(k_i|k_j),$$

(2.19)
where \( \omega'_j \) is defined by

\[
\omega'_j = \begin{cases} 
\frac{2}{\pi} \frac{k_j^2 \omega_j}{k_j^2 - k_0^2} & \text{for } j \leq N, \\
-\frac{2}{\pi} \sum_{m=1}^{N} \frac{\omega_m}{k_m^2 - k_0^2} k_0^2 & \text{for } j = N+1.
\end{cases}
\]  

(2.20)

The matrix \( F \) is nonsingular since \( k_{N+1} \) is distinct from the grid points; it can therefore be inverted to yield the \( R \)-matrix both on and off the energy shell

\[
R^\alpha(k_i|k_{N+1}) = \sum_{j=1}^{N+1} F^{-1}_L(k_i|k_j) V^\alpha_L(k_j|k_{N+1}).
\]  

(2.21)

The extension to coupled channels is straightforward. One simply combines the points and the label \( L \) to form a larger \( (2N+2) \times (2N+2) \) dimensional matrix

\[
V^\alpha(i, j) = \sum_{j=1}^{2N+2} F^\alpha(i, j) R^\alpha(j, j).
\]  

(2.22)

Here the \( i, j, \) and \( j \) labels include both the points \( k_1, \ldots, k_{N+1} \) and the \( L \)-value. For example, for \( L = J - 1 \) we take the \( 1 \leq i \leq N+1 \) points as \( k_1, \ldots, k_{N+1} \). For \( L = J + 1 \), the label \( i \) ranges as \( N+2 \leq i \leq 2N+2 \) with identical \( k \)-values, \( k_1, \ldots, k_{N+1} \equiv k_{N+2}, \ldots, k_{2N+2} \).

\[
R^\alpha_L(k|k_0) = V^\alpha_L(k|k_0) - \frac{2}{\pi} \int_0^\infty dk' \frac{k'^2 V^\alpha_L(k|k') R^\alpha_L(k'|k_0) - k_0^2 V^\alpha_L(k|k_0) R^\alpha_L(k_0|k_0)}{k'^2 - k_0^2}.
\]  

(2.22)
Scattering matrix :  \[ S = 1 - 2 \pi i T \]

**T- Matrix**  \[ T = \frac{K}{1 - iK} \]

**Phase-shift**

For uncoupled channels S- matrix :  \[ S_j = e^{2i\delta_j} \]

Phase-shift in terms of K-matrix :  \[ \delta_j^{0(1)} = \arctan \left( \frac{\pi}{2} q_{\mu 12} K_j^{0(1)}(q_{on}, q_{on}) \right) \]

**Cross-section**

\[ \sigma_l = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2(\delta_l) \]
For coupled channel

For coupled channels the Blatt-Biedenharn eigenphase shifts [5] can be related to the on-shell $R$-matrix elements through the relations [18]

\[
\delta^J_+ = \arctan \left( -q_+ \frac{\mu_+}{4\pi} \left( -R^J_+ + \frac{R^J_+ - ++R^J}{\cos(c^J)} \right) \right) \quad (2.66)
\]

\[
\delta^J_- = \arctan \left( -q_- \frac{\mu_-}{4\pi} \left( -R^J_- + \frac{R^J_- - ++R^J}{\cos(c^J)} \right) \right) \quad (2.67)
\]

\[
e^J = \frac{1}{2} \arctan \left( -2 \frac{+R^J_-}{-R^J_- + ++R^J} \right) \quad (2.68)
\]

Another convention for the phase shifts of coupled channels are the bar-phase shifts or Stapp phase shifts [26]. They are related to the Blatt-Biedenharn eigenphase shifts as

\[
\bar{\epsilon} = \frac{1}{2} \arcsin \left[ \sin (\delta_+ - \delta_-) \sin (2\epsilon) \right] \quad (2.69)
\]

\[
\bar{\delta}_1 = \frac{1}{2} \left[ \delta_+ + \delta_- + \arcsin \left( \frac{\tan (2\bar{\epsilon})}{\tan (2\epsilon)} \right) \right] \quad (2.70)
\]

\[
\bar{\delta}_2 = \delta_+ + \delta_- - \bar{\delta}_1. \quad (2.71)
\]
Low-energy parameters

An important and convenient measure of the interaction is obtained from the effective-range (ER) expansion. For \( q \to 0 \) the \( S \)-wave \( R \)-matrix elements behave as \( \frac{\tan 0 \delta^0}{q} \), and can be expanded as

\[
\frac{q}{\tan 0 \delta^0} = q \cot 0 \delta^0 \approx -\frac{1}{a_s} + \frac{1}{2} r_s q^2
\]  

(2.79)

with the low-energy (LE) parameters \( a_s \) and \( r_s \), the scattering length and the effective range. \( a_s \) is positive, if a bound state exists and negative, if that is not the case. The relation of the low energy parameters to the cross section is \( [25] \)

\[
\lim_{q\to 0^+} \sigma = 4\pi a^2 + \mathcal{O}(k^2).
\]  

(2.80)

The cross section for low momenta thus is dominated by the scattering length.

The LE parameters are determined from the phase shifts by applying the method of least squares (described in more detail e.g. in \( [6] \)) in an interval of about \( q = 0.1 - 1.0 \) MeV with the polynomial

\[
f(x) = \sum_n z_n x^n, \quad x = q^2, \quad n = 0, ..., N.
\]  

(2.81)

\( N \) has to be chosen depending on the desired precision, but should be 1 or greater to include \( r_s \).